

# Geometric approach to the local Jacquet-Langlands correspondence

Yoichi Mieda

**ABSTRACT.** In this paper, we give a purely geometric approach to the local Jacquet-Langlands correspondence for  $\mathrm{GL}(n)$  over a  $p$ -adic field, under the assumption that the invariant of the division algebra is  $1/n$ . We use the  $\ell$ -adic étale cohomology of the Drinfeld tower to construct the correspondence at the level of the Grothendieck groups with rational coefficients. Moreover, assuming that  $n$  is prime, we prove that this correspondence preserves irreducible representations. This gives a purely local proof of the local Jacquet-Langlands correspondence in this case. We need neither a global automorphic technique nor detailed classification of supercuspidal representations of  $\mathrm{GL}(n)$ .

## 1 Introduction

Let  $F$  be a  $p$ -adic field, i.e., a finite extension of  $\mathbb{Q}_p$ . Let  $n \geq 1$  be an integer and  $D$  a central division algebra over  $F$  such that  $\dim_F D = n^2$ . The famous local Jacquet-Langlands correspondence gives a natural bijective correspondence between irreducible discrete series representations of  $\mathrm{GL}_n(F)$  and irreducible smooth representations of  $D^\times$ . Let us recall its precise statement. Write  $\mathbf{Irr}(D^\times)$  for the set of isomorphism classes of irreducible smooth representations of  $D^\times$ . We denote by  $\mathbf{Disc}(\mathrm{GL}_n(F))$  the set of isomorphism classes of irreducible discrete series representations of  $\mathrm{GL}_n(F)$ . For  $\rho \in \mathbf{Irr}(D^\times)$  (resp.  $\pi \in \mathbf{Disc}(\mathrm{GL}_n(F))$ ), we denote the character of  $\rho$  (resp.  $\pi$ ) by  $\theta_\rho$  (resp.  $\theta_\pi$ ). Here  $\theta_\rho$  is a locally constant function on  $D^\times$ , and  $\theta_\pi$  is a locally integrable function on  $\mathrm{GL}_n(F)$  which is locally constant on  $\mathrm{GL}_n(F)^{\mathrm{reg}}$ , the set of regular elements of  $\mathrm{GL}_n(F)$ . The precise statement of the local Jacquet-Langlands correspondence is the following:

**Theorem 1.1 (the local Jacquet-Langlands correspondence)** *There exists a unique bijection*

$$JL: \mathbf{Irr}(D^\times) \xrightarrow{\cong} \mathbf{Disc}(\mathrm{GL}_n(F))$$

---

Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395 Japan  
 E-mail address: [mieda@math.kyushu-u.ac.jp](mailto:mieda@math.kyushu-u.ac.jp)

2010 *Mathematics Subject Classification.* Primary: 22E50; Secondary: 11F70, 14G35.

satisfying the following character relation: for every regular element  $h$  of  $D^\times$ ,  $\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h)$ , where  $g_h$  is an arbitrary element of  $\mathrm{GL}_n(F)$  whose minimal polynomial is the same as that of  $h$ .

The original proof of this theorem, due to Deligne-Kazhdan-Vigneras [DKV84] and Rogawski [Rog83], was accomplished by using a global automorphic method. In some cases, more explicit local studies can be found in [Hen93], [BH00], [BH05], which are based on the theory of types. However, apart from the case of  $\mathrm{GL}(2)$ , a purely local proof of Theorem 1.1 seems not to be known yet (*cf.* [Hen06, a comment after Theorem 2]).

In this article, under the assumption that the invariant of  $D$  is  $1/n$ , we will give a geometric approach to construct the bijection  $JL$  above. Let  $R(D^\times)$  be the Grothendieck group of finite-dimensional smooth representations of  $D^\times$ , and  $\overline{R}(\mathrm{GL}_n(F))$  the Grothendieck group of finite length smooth representations of  $\mathrm{GL}_n(F)$  “modulo induced representations” (for a precise definition, see [Kaz86]). It is known that the classes of elements of  $\mathrm{Irr}(D^\times)$  (resp.  $\mathrm{Disc}(\mathrm{GL}_n(F))$ ) form a basis of  $R(D^\times)$  (resp.  $\overline{R}(\mathrm{GL}_n(F))$ ). Put  $R(D^\times)_{\mathbb{Q}} = R(D^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\overline{R}(\mathrm{GL}_n(F))_{\mathbb{Q}} = \overline{R}(\mathrm{GL}_n(F)) \otimes_{\mathbb{Z}} \mathbb{Q}$ . The main theorems of this article are the following:

**Theorem 1.2 (Theorem 6.6)** *We can construct the following two homomorphisms geometrically:*

$$JL: R(D^\times)_{\mathbb{Q}} \longrightarrow \overline{R}(\mathrm{GL}_n(F))_{\mathbb{Q}}, \quad LJ: \overline{R}(\mathrm{GL}_n(F))_{\mathbb{Q}} \longrightarrow R(D^\times)_{\mathbb{Q}}.$$

These two maps are inverse to each other, and satisfy the character relations

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every regular  $h \in D^\times$ .

**Theorem 1.3 (Theorem 6.10)** *If  $n$  is prime, then  $JL$  induces a bijection*

$$JL: \mathrm{Irr}(D^\times) \xrightarrow{\cong} \mathrm{Disc}(\mathrm{GL}_n(F)).$$

Theorem 1.3 provides a purely local proof of Theorem 1.1 in the case above. In particular, the local Jacquet-Langlands correspondence for  $\mathrm{GL}_2(F)$  and  $\mathrm{GL}_3(F)$  are fully recovered.

The geometric object we use is the Drinfeld tower for  $\mathrm{GL}_n(F)$ . It is a tower of rigid spaces over (a disjoint union of) the  $(n-1)$ -dimensional Drinfeld upper half space  $\mathbb{P}_F^{n-1} \setminus \bigcup_H H$ , where  $H$  runs through hyperplanes of  $\mathbb{P}_F^{n-1}$  defined over  $F$  (for more detailed explanation, see Section 2). Thanks to extensive studies by many people (*cf.* [Har97], [HT01], [Boy09], [Dat07]), it is now well-known that the local Jacquet-Langlands correspondence is realized in the  $\ell$ -adic cohomology  $H_{\mathrm{Dr}}$  of the Drinfeld tower. The methods in the works cited above are again global and automorphic. However, there is also a purely local study of the cohomology due

to Faltings [Fal94]. He began with an irreducible smooth representation  $\rho$  of  $D^\times$ , and investigated the  $\rho$ -isotypic part  $H_{\text{Dr}}[\rho]$  of  $H_{\text{Dr}}$  by means of the Lefschetz trace formula. By this method, he succeeded to observe that the character relation in Theorem 1.1 appears naturally in  $H_{\text{Dr}}[\rho]$ . By using this result, we can give the map  $JL$  in Theorem 1.2.

To construct the inverse map  $LJ$ , we need to consider the opposite direction; we begin with an irreducible discrete series representation  $\pi$  of  $\text{GL}_n(F)$  and investigate the “ $\pi$ -isotypic part” of  $H_{\text{Dr}}$ . More precisely, we should consider the alternating sum of the extension groups  $\sum_j (-1)^j \text{Ext}^j(H_{\text{Dr}}, \pi)$ , since  $\pi$  is neither projective nor injective in general. To study it, we apply the method introduced in [Mie11]; namely, we use local harmonic analysis, such as transfer of orbital integrals. Furthermore, to prove Theorem 1.3, the non-cuspidality result obtained in [Mie10b] plays a crucial role.

Since our approach is entirely geometric, it is natural to expect that a similar argument may give interesting consequences for the mod- $\ell$  Jacquet-Langlands correspondence (*cf.* [Dat11]). The author also expects that our strategy can be extended to other Rapoport-Zink spaces, especially the Rapoport-Zink space for  $\text{GSp}(4)$ . He hopes to deal with these problems in future works.

We sketch the outline of this paper. In Section 2, we recall the definition of the Drinfeld tower and results in [Fal94]. We use the framework of [Dat00] to deal with finitely generated representations systematically. In Section 3, we study the alternating sum of the extension groups  $\sum_j (-1)^j \text{Ext}^j(H_{\text{Dr}}, \pi)$  by means of local harmonic analysis. In Section 4, we apply another deep result of Faltings on the comparison of the Lubin-Tate tower and the Drinfeld tower. It provides a very important finiteness result on  $H_{\text{Dr}}$ . Under this finiteness, results in Section 2 and Section 3 can be written in a very simple form. After short preliminaries on representation theory in Section 5, finally in Section 6, we construct the maps  $JL$  and  $LJ$  in Theorem 1.2 by using the  $\ell$ -adic cohomology of the Drinfeld tower, and investigate their properties.

**Acknowledgment** The author would like to thank Matthias Strauch for very helpful discussions. He is also grateful to Tetsushi Ito and Sug Woo Shin for their valuable comments.

**Notation** For a totally disconnected locally compact group  $H$ , let  $\text{Irr}(H)$  be the set of isomorphism classes of irreducible smooth representations of  $H$ . We denote the Grothendieck group of finitely generated (resp. finite length) smooth  $H$ -representations by  $K(H)$  (resp.  $R(H)$ ). Put  $R(H)_{\mathbb{Q}} = R(H) \otimes_{\mathbb{Z}} \mathbb{Q}$ . For a finite-dimensional smooth representation  $\sigma$  of  $H$ , write  $\theta_\sigma$  for the character of  $\sigma$ . It is a locally constant function on  $H$ . If moreover a Haar measure on  $H$  is fixed, we denote by  $\mathcal{H}(H)$  the Hecke algebra of  $H$ , namely, the abelian group of locally constant compactly supported functions on  $H$  with convolution product. Put  $\overline{\mathcal{H}}(H) = \mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)] = \mathcal{H}(H)_H$  (the  $H$ -coinvariant quotient).

Let  $F$  be a  $p$ -adic field and  $\mathcal{O}$  its ring of integers. We denote the normalized valuation of  $F$  by  $v_F$  and the cardinality of the residue field of  $\mathcal{O}$  by  $q$ . Fix a uniformizer  $\varpi$  of  $\mathcal{O}$ . Denote the completion of the maximal unramified extension of  $\mathcal{O}$  by  $\check{\mathcal{O}}$  and the fraction field of  $\check{\mathcal{O}}$  by  $\check{F}$ .

Throughout this paper, we fix an integer  $n \geq 1$ . Let  $D$  be the central division algebra over  $F$  with invariant  $1/n$ , and  $\mathcal{O}_D$  its maximal order. Fix a uniformizer  $\Pi \in \mathcal{O}_D$  such that  $\Pi^n = \varpi$ .

For simplicity, put  $G = \mathrm{GL}_n(F)$ . We denote by  $G^{\mathrm{reg}}$  (resp.  $G^{\mathrm{ell}}$ ) the set of regular (resp. regular elliptic) elements of  $G$ . Write  $Z_G$  for the center of  $G$ . We apply these notations to other groups. For example, we write  $(D^\times)^{\mathrm{reg}}$  for the set of regular elements of  $D^\times$ . As in Theorem 1.1, for  $h \in (D^\times)^{\mathrm{reg}}$ , let  $g_h$  be an element of  $G^{\mathrm{ell}}$  whose minimal polynomial is the same as that of  $h$ . Such an element always exists, and is unique up to conjugacy. Moreover,  $h \mapsto g_h$  induces a bijection between conjugacy classes in  $(D^\times)^{\mathrm{reg}}$  and those in  $G^{\mathrm{ell}}$ . Therefore, to  $g \in G^{\mathrm{ell}}$  we can attach an element  $h_g \in (D^\times)^{\mathrm{reg}}$  whose minimal polynomial is the same as that of  $g$ .

For a smooth  $G$ -representation  $\pi$  of finite length, we denote by  $\theta_\pi$  the distribution character of  $\pi$ . It is a locally integrable function on  $G$  which is locally constant on  $G^{\mathrm{reg}}$ .

We identify  $F^\times$  with  $Z_G$  and  $Z_{D^\times}$ . Then, we can consider the quotient groups  $G/\varpi^\mathbb{Z}$  and  $D^\times/\varpi^\mathbb{Z}$  under a discrete subgroup  $\varpi^\mathbb{Z}$  of  $F^\times$ . We regard  $\mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$  (resp.  $\mathbf{Irr}(G/\varpi^\mathbb{Z})$ ) as a subset of  $\mathbf{Irr}(D^\times)$  (resp.  $\mathbf{Irr}(G)$ ). Similarly,  $R(D^\times/\varpi^\mathbb{Z})$  (resp.  $R(G/\varpi^\mathbb{Z})$ ) is regarded as a submodule of  $R(D^\times)$  (resp.  $R(G)$ ). Write  $\mathbf{Disc}(G)$  for the set of isomorphism classes of irreducible discrete series representations of  $G$ . Put  $\mathbf{Disc}(G/\varpi^\mathbb{Z}) = \mathbf{Disc}(G) \cap \mathbf{Irr}(G/\varpi^\mathbb{Z})$ .

Fix Haar measures on  $G$  and  $D^\times$ . We endow  $\varpi^\mathbb{Z}$  with the counting measure and consider the quotient measures on  $G/\varpi^\mathbb{Z}$  and  $D^\times/\varpi^\mathbb{Z}$ . For  $\varphi \in \mathcal{H}(G/\varpi^\mathbb{Z})$  and  $g \in G^{\mathrm{ell}}$ , put  $O_g^{G/\varpi^\mathbb{Z}}(\varphi) = \int_{G/\varpi^\mathbb{Z}} \varphi(x^{-1}gx)dx$  (the orbital integral). It is well-known that this integral converges. Similarly, for  $\varphi' \in \mathcal{H}(D^\times/\varpi^\mathbb{Z})$  and  $h \in D^\times$ , put  $O_h^{D^\times/\varpi^\mathbb{Z}}(\varphi') = \int_{D^\times/\varpi^\mathbb{Z}} \varphi'(y^{-1}hy)dy$ .

For a field  $k$ , we denote its algebraic closure by  $\bar{k}$ . Let  $\ell$  be a prime which is invertible in  $\mathcal{O}$ . We fix an isomorphism  $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$  and identify them. Every representation is considered over  $\mathbb{C}$ .

## 2 Drinfeld tower

Let us briefly recall the definition of the Drinfeld tower. For more detailed description, see [Dri76], [BC91], [RZ96, Chapter 3].

First of all, fix a special formal  $\mathcal{O}_D$ -module  $\mathbb{X}$  of  $\mathcal{O}_F$ -height  $n^2$  over  $\bar{\mathbb{F}}_q$ . It is well-known that such  $\mathbb{X}$  is unique up to  $\mathcal{O}_D$ -isogeny.

We denote by  $\mathbf{Nilp}$  the category of  $\check{\mathcal{O}}$ -schemes on which  $\varpi$  is locally nilpotent. Consider the functor  $\mathcal{M}_{\mathrm{Dr}}$  from  $\mathbf{Nilp}$  to the category of sets that maps  $S$  to the set of isomorphism classes of pairs  $(X, \rho)$  consisting of

- a special formal  $\mathcal{O}_D$ -module  $X$  over  $S$ ,
- and an  $\mathcal{O}_D$ -quasi-isogeny  $\rho: \mathbb{X} \otimes_{\mathbb{F}_q} \overline{S} \longrightarrow X \otimes_S \overline{S}$ ,

where we put  $\overline{S} = S \otimes_{\mathcal{O}} \overline{\mathbb{F}}_q$ . Then  $\mathcal{M}_{\text{Dr}}$  is represented by a formal scheme locally of finite type over  $\check{\mathcal{O}}$ . We denote the formal scheme by  $\mathcal{M}_{\text{Dr}}$  again, and the rigid generic fiber of  $\mathcal{M}_{\text{Dr}}$  by  $M_{\text{Dr}}$ . It is known that  $M_{\text{Dr}}$  is the disjoint union of countable copies of the  $(n-1)$ -dimensional Drinfeld upper half space.

For an integer  $m \geq 0$ , let  $M_{\text{Dr},m}$  be the rigid space classifying  $\Pi^m$ -level structures on the universal formal  $\mathcal{O}_D$ -module over  $M_{\text{Dr}}$ . It is a finite étale Galois covering of  $M_{\text{Dr}}$  with Galois group  $(\mathcal{O}_D/(\Pi^m))^{\times}$ . The projective system  $\{M_{\text{Dr},m}\}_{m \geq 0}$  is called the Drinfeld tower. We can define a natural right action of  $G = \text{GL}_n(F)$  on each  $M_{\text{Dr},m}$  because  $G$  is isomorphic to the group of self  $\mathcal{O}_D$ -quasi-isogenies of  $\mathbb{X}$ . On the other hand,  $D^{\times}$  also acts naturally on  $M_{\text{Dr},m}$  on the right, since  $1 + \Pi^m \mathcal{O}_D$  is a normal subgroup of  $D^{\times}$ .

Since  $M_m$  is too large (it has infinitely many connected components), we take the quotient  $M_{\text{Dr},m}/\varpi^{\mathbb{Z}}$  of  $M_{\text{Dr},m}$  by  $\varpi^{\mathbb{Z}} \subset F^{\times} = Z_G \subset G$ . Put

$$H_{\text{Dr},m}^i = H_c^i((M_{\text{Dr},m}/\varpi^{\mathbb{Z}}) \otimes_{\check{F}} \overline{\mathbb{F}}, \overline{\mathbb{Q}}_{\ell}), \quad H_{\text{Dr}}^i = \varinjlim_m H_{\text{Dr},m}^i.$$

These are smooth representations of  $G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}$ . Unless  $n-1 \leq i \leq 2(n-1)$ ,  $H_{\text{Dr},m}^i = H_{\text{Dr}}^i = 0$ .

**Proposition 2.1** *The representation  $H_{\text{Dr},m}^i$  is finitely generated as a  $G$ -module. Moreover, there exist compact open subgroups  $K_1, \dots, K_N$  of  $G/\varpi^{\mathbb{Z}}$ ,  $\varepsilon_{\nu} \in \{\pm 1\}$  and finite-dimensional smooth representations  $\sigma_{m,\nu}$  of  $K_{\nu} \times D^{\times}/\varpi^{\mathbb{Z}}$  for each  $1 \leq \nu \leq N$  such that the following holds:*

$$\sum_i (-1)^i [H_{\text{Dr},m}^i] = \sum_{\nu=1}^N \varepsilon_{\nu} [\text{c-Ind}_{K_{\nu}}^{G/\varpi^{\mathbb{Z}}} \sigma_{m,\nu}] \quad \text{in } K(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}).$$

*Proof.* We only give a sketch of a proof, since it seems to be well-known (cf. [Far04, Proposition 4.4.13]).

It is known that  $M_{\text{Dr}}/\varpi^{\mathbb{Z}}$  has an open covering  $\mathfrak{U} = \{U_{\lambda}\}_{\lambda \in \Lambda}$  consisting of quasi-compact open subsets, indexed by the set  $\Lambda$  of vertices of the Bruhat-Tits building of  $\text{PGL}_n(F)$ . This covering satisfies the following properties:

- (a) For  $g \in G/\varpi^{\mathbb{Z}}$ ,  $U_{\lambda} \cdot g = U_{g^{-1}\lambda}$ .
- (b) For each  $\lambda \in \Lambda$ , there exist only finitely many  $\lambda' \in \Lambda$  satisfying  $U_{\lambda} \cap U_{\lambda'} \neq \emptyset$ .
- (c) For each  $\lambda \in \Lambda$ ,  $K_{\lambda} = \{g \in G/\varpi^{\mathbb{Z}} \mid U_{\lambda} \cdot g = U_{\lambda}\}$  is a compact open subgroup of  $G/\varpi^{\mathbb{Z}}$ .

For a finite subset  $I \subset \Lambda$ , put

$$U_I = \bigcap_{\lambda \in I} U_{\lambda}, \quad K_I = \{g \in G/\varpi^{\mathbb{Z}} \mid U_I \cdot g = U_I\}.$$

For an integer  $r \geq 0$ , set  $\Lambda_r = \{I \subset \Lambda \mid \#I = r+1, U_I \neq \emptyset\}$ . Then, by the three properties above, we have the following:

- For each  $r \geq 0$ , the set of  $G/\varpi^{\mathbb{Z}}$ -orbits in  $\Lambda_r$  is finite.
- We have  $\Lambda_r = \emptyset$  for sufficiently large  $r$ .
- For every finite subset  $I \subset \Lambda$ ,  $K_I$  is a compact open subgroup of  $G/\varpi^{\mathbb{Z}}$ .

Take a system of representatives  $I_{r,1}, \dots, I_{r,N_r}$  of  $(G/\varpi^{\mathbb{Z}}) \setminus \Lambda_r$  and put  $K_{r,i} = K_{I_{r,i}}$ .

Let  $\mathfrak{U}_m = \{U_{m,\lambda}\}_{\lambda \in \Lambda}$  be the covering obtained as the inverse image of  $\mathfrak{U}$ . For a finite subset  $I \subset \Lambda$ , put  $U_{m,I} = \bigcap_{\lambda \in I} U_{m,\lambda}$ . Then we have the Čech spectral sequence

$$E_1^{-r,s} = \bigoplus_{I \in \Lambda_r} H_c^s(U_{m,I} \otimes_{\check{F}} \overline{\check{F}}, \overline{\mathbb{Q}}_{\ell}) \Longrightarrow H_{\text{Dr},m}^{-r+s}.$$

Put

$$V_{m,r,i}^s = H_c^s(U_{m,I_{r,i}} \otimes_{\check{F}} \overline{\check{F}}, \overline{\mathbb{Q}}_{\ell}).$$

It is a finite-dimensional smooth representation of  $K_{r,i} \times D^{\times}/\varpi^{\mathbb{Z}}$  and vanishes for  $s > 2n$  (cf. [Hub96, Proposition 5.5.1, Proposition 6.3.2]). We can easily observe that  $E_1^{-r,s}$  is isomorphic to  $\bigoplus_{i=1}^{N_r} \text{c-Ind}_{K_{r,i}}^{G/\varpi^{\mathbb{Z}}} V_{m,r,i}^s$  as a  $G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}$ -representation. Therefore  $E_1^{-r,s}$  is a finitely generated  $G/\varpi^{\mathbb{Z}}$ -representation, and vanishes for all but finitely many  $(r, s)$ . Hence  $H_{\text{Dr},m}^i$  is finitely generated as a  $G$ -module (cf. [Ber84, Remarque 3.12]). Moreover, in  $K(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}})$  we have

$$\sum_i (-1)^i [H_{\text{Dr},m}^i] = \sum_{r,s} (-1)^{-r+s} E_1^{-r,s} = \sum_{r,s} \sum_{i=1}^{N_r} (-1)^{-r+s} \text{c-Ind}_{K_{r,i}}^{G/\varpi^{\mathbb{Z}}} V_{m,r,i}^s.$$

This concludes the proof. ■

**Definition 2.2** We denote by  $\eta_m$  the image of  $\sum_i (-1)^i [H_{\text{Dr},m}^i]$  under the rank map

$$\text{Rk}: K(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}) \longrightarrow \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}})$$

(cf. [Dat00, 1.2]). For  $h \in D^{\times}$ , we define  $\eta_{m,h} \in \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}})$  by

$$\eta_{m,h}(g) = \int_{D^{\times}/\varpi^{\mathbb{Z}}} \eta_m(g, h'^{-1}hh') dh'.$$

Using the expression of  $\sum_i (-1)^i [H_{\text{Dr},m}^i]$  in Proposition 2.1, we can give more explicit description of  $\eta_m$  and  $\eta_{m,h}$ :

**Proposition 2.3** For  $m \geq 0$  and  $h \in D^{\times}$ , define  $\tilde{\eta}_m \in \mathcal{H}(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}})$  and  $\tilde{\eta}_{m,h} \in \mathcal{H}(G/\varpi^{\mathbb{Z}})$  by

$$\tilde{\eta}_m = \sum_{\nu=1}^N \frac{\varepsilon_{\nu} \theta_{\sigma_{m,\nu}^{\vee}}}{\text{vol}(K_{\nu} \times D^{\times}/\varpi^{\mathbb{Z}})}, \quad \tilde{\eta}_{m,h} = \sum_{\nu=1}^N \frac{\varepsilon_{\nu} \theta_{\sigma_{m,\nu}^{\vee}}(-, h)}{\text{vol}(K_{\nu})},$$

where  $(-)^{\vee}$  denotes the contragredient, and  $\theta_{\sigma_{m,\nu}^{\vee}}$  (resp.  $\theta_{\sigma_{m,\nu}^{\vee}}(-, h)$ ) is regarded as a function on  $G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}$  (resp.  $G/\varpi^{\mathbb{Z}}$ ) by setting  $\theta_{\sigma_{m,\nu}^{\vee}}(g, h) = 0$  for  $g \notin K_{\nu}$ .

Then, the image of  $\tilde{\eta}_m$  (resp.  $\tilde{\eta}_{m,h}$ ) in  $\overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}})$  (resp.  $\overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}})$ ) coincides with  $\eta_m$  (resp.  $\eta_{m,h}$ ).

*Proof.* The assertion for  $\tilde{\eta}_{m,h}$  immediately follows from that for  $\tilde{\eta}_m$ . Thus it suffices to prove the following:

Let  $K$  be a compact open subgroup of  $G/\varpi^{\mathbb{Z}}$ . For every finite-dimensional smooth representation  $\sigma$  of  $K \times D^{\times}/\varpi^{\mathbb{Z}}$ , the image of  $\text{vol}(K)^{-1}\theta_{\sigma^{\vee}}$  in  $\overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}})$  coincides with  $\text{Rk}([\text{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma])$ .

Since the image of  $[\sigma]$  under the rank map  $\text{Rk}: K(K \times D^{\times}/\varpi^{\mathbb{Z}}) \longrightarrow \overline{\mathcal{H}}(K \times D^{\times}/\varpi^{\mathbb{Z}})$  is  $\text{vol}(K \times D^{\times}/\varpi^{\mathbb{Z}})^{-1}\theta_{\sigma^{\vee}}$ , this claim follows from the commutative diagram below (cf. [Dat00, proof of Theorem 1.6]):

$$\begin{array}{ccc} K(K \times D^{\times}/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(K \times D^{\times}/\varpi^{\mathbb{Z}}) \\ \text{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \downarrow & & \downarrow \text{extention by 0} \\ K(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}). \end{array} \quad \blacksquare$$

In [Fal94], Faltings investigated the function  $\tilde{\eta}_m$  above by means of the Lefschetz trace formula. His results can be summarized in the following theorem.

**Theorem 2.4** *Let  $g \in G^{\text{reg}}$  and  $h \in D^{\times}$ .*

i) *If  $g$  is elliptic, then we have*

$$\begin{aligned} O_g^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) &= \# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\varpi^{\mathbb{Z}}) \\ &= n \cdot \#\{a \in D^{\times}/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D) \mid hah_g^{-1} = a\}. \end{aligned}$$

ii) *If  $g$  is not elliptic, then we have  $\int_{Z(g) \backslash G} \eta_{m,h}(x^{-1}gx) dx = 0$ , where  $Z(g)$  denotes the centralizer of  $g$ .*

*Proof.* Let us briefly recall the proof in [Fal94]. We use the notation in the proof of Proposition 2.1.

First consider the case where  $g$  is elliptic. Then, we can find a finite subset  $\Lambda_g \subset \Lambda$  such that  $g\Lambda_g = \Lambda_g$  and  $gU_{\lambda} \cap U_{\lambda} = \emptyset$  for  $\lambda \in \Lambda \setminus \Lambda_g$ . Put  $U_{m,g} = \bigcup_{\lambda \in \Lambda_g} U_{m,\lambda}$ . Then  $U_{m,g}$  is quasi-compact smooth and  $(g^{-1}, h^{-1}): U_{m,g} \longrightarrow U_{m,g}$  has no fixed point on the boundary of  $U_{m,g}$ . Therefore we can apply the Lefschetz trace formula for this endomorphism (for a general theory of the Lefschetz trace formula for rigid spaces, see [Mie10a]). Noting that every fixed point of  $(g^{-1}, h^{-1}): M_{\text{Dr},m}/\varpi^{\mathbb{Z}} \longrightarrow M_{\text{Dr},m}/\varpi^{\mathbb{Z}}$  lies in  $U_{m,g}$ , we obtain the following equality:

$$\sum_i (-1)^i \text{Tr}((g^{-1}, h^{-1}); H_c^i(U_{m,g} \otimes_{\breve{F}} \overline{\mathbb{Q}}_{\ell})) = \# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\varpi^{\mathbb{Z}}).$$

By the Čech spectral sequence, we can easily show that the left hand side is equal to  $O_g^{G/\varpi^{\mathbb{Z}}}(\tilde{\eta}_{m,h})$ . The right hand side can be computed by using the period map, as in [Str08, §2.6]. The result is

$$\# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\varpi^{\mathbb{Z}}) = n \cdot \#\{a \in D^{\times}/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D) \mid hah_g^{-1} = a\}.$$

It is slightly different from [Fal94, Theorem 1], because our  $M_{\text{Dr}}/\varpi^{\mathbb{Z}}$  is the disjoint union of  $n$  copies of  $\Omega$  considered in [Fal94]. Rather, it is compatible with [Str08, Theorem 2.6.8]. This concludes the proof of i).

To prove ii), apply the same argument to  $M_{\text{Dr},m}/\Gamma$ , where  $\Gamma$  is a sufficiently small discrete torsion-free cocompact subgroup of  $Z(g)$ .  $\blacksquare$

**Corollary 2.5** *For  $\rho \in \text{Irr}(D^{\times}/\varpi^{\mathbb{Z}})$ ,  $\text{Hom}_{D^{\times}}(\rho, H_{\text{Dr}}^i)$  is a finitely generated  $G/\varpi^{\mathbb{Z}}$ -representation by [Mie11, Lemma 5.2]. The image of  $\sum_i (-1)^i [\text{Hom}_{D^{\times}}(\rho, H_{\text{Dr}}^i)]$  under the map*

$$\text{Rk}^{\vee}: K(G/\varpi^{\mathbb{Z}}) \xrightarrow{\text{Rk}} \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}}) \xrightarrow{\vee} C^{\infty}(G^{\text{ell}})$$

*coincides with  $g \mapsto n\theta_{\rho}(h_g^{-1})$ . Recall that for  $f \in \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}})$  the locally constant function  $f^{\vee}$  on  $G^{\text{ell}}$  is given by  $f^{\vee}(g) = \int_{G/\varpi^{\mathbb{Z}}} f(xg^{-1}x^{-1})dx$  (cf. [Dat00, p. 190]).*

*Proof.* Take a sufficiently large integer  $m \geq 0$  so that  $\rho|_{1+\Pi^m \mathcal{O}_D}$  is trivial. Then we have  $\text{Hom}_{D^{\times}}(\rho, H_{\text{Dr}}^i) = \text{Hom}_{D^{\times}}(\rho, (H_{\text{Dr}}^i)^{1+\Pi^m \mathcal{O}_D}) = \text{Hom}_{D^{\times}}(\rho, H_{\text{Dr},m}^i)$ .

It is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} K(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^{\times}/\varpi^{\mathbb{Z}}) \\ \text{Hom}_{D^{\times}}(\rho, -) \downarrow & & \downarrow (*) \\ K(G/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}}), \end{array}$$

where  $(*)$  is given by

$$f \mapsto \left( g \mapsto \int_{D^{\times}/\varpi^{\mathbb{Z}}} f(g, h) \theta_{\rho}(h) dh \right).$$

Therefore, the image of  $\sum_i (-1)^i [\text{Hom}_{D^{\times}}(\rho, H_{\text{Dr},m}^i)]$  under  $\text{Rk}^{\vee}$  can be calculated as follows:

$$\begin{aligned} g &\mapsto \int_{G/\varpi^{\mathbb{Z}}} \int_{D^{\times}/\varpi^{\mathbb{Z}}} \eta_m(xg^{-1}x^{-1}, h) \theta_{\rho}(h) dh dx \\ &= \frac{1}{\text{vol}(D^{\times}/\varpi^{\mathbb{Z}})} \int_{G/\varpi^{\mathbb{Z}}} \int_{D^{\times}/\varpi^{\mathbb{Z}}} \int_{D^{\times}/\varpi^{\mathbb{Z}}} \eta_m(xg^{-1}x^{-1}, h) \theta_{\rho}(h'hh'^{-1}) dh' dh dx \\ &= \frac{1}{\text{vol}(D^{\times}/\varpi^{\mathbb{Z}})} \int_{D^{\times}/\varpi^{\mathbb{Z}}} O_{g^{-1}}^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) \theta_{\rho}(h) dh \\ &= \frac{n}{\text{vol}(D^{\times}/\varpi^{\mathbb{Z}})} \int_{D^{\times}/\varpi^{\mathbb{Z}}} \#\{a \in D^{\times}/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D) \mid hah_g = a\} \theta_{\rho}(h) dh \\ &= \frac{n}{\#(D^{\times}/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D))} \\ &\quad \times \sum_{h \in D^{\times}/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D)} \#\{a \in D^{\times}/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D) \mid hah_g = a\} \theta_{\rho}(h) \end{aligned}$$

$$\begin{aligned}
 &= \frac{n}{\#(D^\times/\varpi^\mathbb{Z}(1 + \Pi^m \mathcal{O}_D))} \sum_{a \in D^\times/\varpi^\mathbb{Z}(1 + \Pi^m \mathcal{O}_D))} \theta_\rho(ah_g^{-1}a^{-1}) \\
 &= n\theta_\rho(h_g^{-1}).
 \end{aligned}$$

This completes the proof. ■

### 3 Some harmonic analysis

In this section, we use Theorem 2.4 to investigate the virtual  $D^\times$ -representation  $\sum_{i,j \geq 0} (-1)^{i+j} \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)$ . In the Lubin-Tate case, a similar study is carried out in [Mie11]. For  $m \geq 0$ , denote by  $K'_m$  the image of  $1 + \Pi^m \mathcal{O}_D$  in  $D^\times/\varpi^\mathbb{Z}$ .

**Lemma 3.1** *For a smooth representation  $V$  of  $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$  and a smooth representation  $\pi$  of  $G/\varpi^\mathbb{Z}$ , we have  $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(V, \pi)^{K'_m} \cong \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V^{K'_m}, \pi)$ . Here  $\text{Ext}_{G/\varpi^\mathbb{Z}}^j$  is taken in the category of smooth  $G/\varpi^\mathbb{Z}$ -representations.*

*Proof.* First we prove that there exist a smooth  $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -representation  $P$  which is projective as a  $G/\varpi^\mathbb{Z}$ -representation and a  $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -equivariant surjection  $P \rightarrow V$ . Take a  $G/\varpi^\mathbb{Z}$ -equivariant surjection  $P' \rightarrow V \rightarrow 0$  from a projective  $G/\varpi^\mathbb{Z}$ -representation  $P'$ . Put  $P = P' \otimes_{\mathbb{C}} C_c^\infty(D^\times/\varpi^\mathbb{Z})$ . Then  $P$  is a smooth  $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -representation which is projective as a  $G/\varpi^\mathbb{Z}$ -representation, and a surjection  $P \rightarrow V$  is naturally induced.

Therefore, we can take a  $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -equivariant resolution  $P_\bullet \rightarrow V \rightarrow 0$  of  $V$  such that  $P_i$  is projective as a smooth  $G/\varpi^\mathbb{Z}$ -representation. Since  $P_i^{K'_m}$  is a direct summand of  $P_i$  as a  $G/\varpi^\mathbb{Z}$ -representation,  $P_i^{K'_m}$  is also projective. Thus  $P_\bullet^{K'_m} \rightarrow V^{K'_m} \rightarrow 0$  gives a projective resolution of  $V^{K'_m}$ . Hence we have

$$\begin{aligned}
 \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V, \pi)^{K'_m} &= H^j(\text{Hom}_{G/\varpi^\mathbb{Z}}(P_\bullet, \pi))^{K'_m} \cong H^j(\text{Hom}_{G/\varpi^\mathbb{Z}}(P_\bullet^{K'_m}, \pi)) \\
 &= \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V^{K'_m}, \pi),
 \end{aligned}$$

as desired. ■

**Corollary 3.2** *For every  $m \geq 0$  and  $\pi \in \text{Irr}(G/\varpi^\mathbb{Z})$ , we have*

$$\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{K'_m} \cong \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi).$$

*It is finite-dimensional and vanishes if  $j \geq n$ . In particular,  $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}$  is an admissible representation of  $D^\times/\varpi^\mathbb{Z}$  and vanishes if  $j \geq n$ , where  $(-)^{\text{sm}}$  denotes the set of  $D^\times/\varpi^\mathbb{Z}$ -smooth vectors.*

*Proof.* Lemma 3.1 tells us that

$$\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{K'_m} = \text{Ext}_{G/\varpi^\mathbb{Z}}^j((H_{\text{Dr}}^i)^{K'_m}, \pi) = \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi).$$

By Proposition 2.1 and [SS97, Corollary II.3.3], it is finite-dimensional and vanishes if  $j \geq n$ . ■

**Remark 3.3** Later (Corollary 4.3) we will prove that  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi)$  is in fact a finite-dimensional smooth  $D^{\times}$ -representation.

The character of  $\sum_{i,j \geq 0} (-1)^{i+j} \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr},m}^i, \pi)$  can be computed by  $\eta_{m,h}$  introduced in the previous section:

**Proposition 3.4** *For every  $\pi \in \mathrm{Irr}(G/\varpi^{\mathbb{Z}})$  and  $h \in D^{\times}/\varpi^{\mathbb{Z}}$ , we have*

$$\sum_{i,j \geq 0} (-1)^{i+j} \mathrm{Tr}(h; \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr},m}^i, \pi)) = \mathrm{Tr}(\eta_{m,h}; \pi) = \int_{G/\varpi^{\mathbb{Z}}} \eta_{m,h}(g) \theta_{\pi}(g) dg.$$

*Proof.* First note that  $V \mapsto \sum_{j \geq 0} (-1)^j \mathrm{Tr}(h, \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(V, \pi))$  induces a homomorphism  $K(G/\varpi^{\mathbb{Z}}) \rightarrow \mathbb{C}$  of abelian groups. Therefore, by Proposition 2.1 and Proposition 2.3, we have only to show the following:

Let  $K$  be a compact open subgroup of  $G/\varpi^{\mathbb{Z}}$ . For every finite-dimensional smooth representation  $\sigma$  of  $K \times D^{\times}/\varpi^{\mathbb{Z}}$  and  $h \in D^{\times}$ , we have

$$\sum_{j \geq 0} (-1)^j \mathrm{Tr}(h; \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(\mathrm{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma, \pi)) = \mathrm{Tr}\left(\frac{\theta_{\sigma^{\vee}}(-, h)}{\mathrm{vol}(K)}; \pi\right).$$

Since  $\mathrm{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma$  is a projective  $G/\varpi^{\mathbb{Z}}$ -representation,  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(\mathrm{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma, \pi) = 0$  for  $j \geq 1$ . Take an open normal subgroup  $K_1 \subset K$  such that  $\sigma|_{K_1}$  is trivial. Then the left hand side can be computed as follows:

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \mathrm{Tr}(h; \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(\mathrm{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma, \pi)) &= \mathrm{Tr}(h; \mathrm{Hom}_{G/\varpi^{\mathbb{Z}}}(\mathrm{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma, \pi)) \\ &= \mathrm{Tr}(h; \mathrm{Hom}_K(\sigma, \pi|_K)) = \mathrm{Tr}(h; \mathrm{Hom}_{K/K_1}(\sigma, \pi^{K_1})) \\ &= \frac{1}{\#(K/K_1)} \sum_{g \in K/K_1} \mathrm{Tr}((g^{-1}, h^{-1}); \sigma) \mathrm{Tr}(g; \pi^{K_1}) \\ &= \frac{1}{\mathrm{vol}(K)} \mathrm{Tr}(\theta_{\sigma^{\vee}}(-, h); \pi). \end{aligned}$$

This completes the proof. ■

**Lemma 3.5** *For every  $g \in G^{\mathrm{ell}}$ ,  $h \in D^{\times}$  and an integer  $m \geq 0$ , we have*

$$O_g^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) = n O_{h_g}^{D^{\times}/\varpi^{\mathbb{Z}}} \left( \frac{\mathbf{1}_{hK'_m}}{\mathrm{vol}(K'_m)} \right),$$

where  $\mathbf{1}_{hK'_m}$  denotes the characteristic function of  $hK'_m$ .

*Proof.* By Theorem 2.4 i), we obtain

$$\begin{aligned} O_g^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) &= n \cdot \# \{a \in (D^{\times}/\varpi^{\mathbb{Z}})/K'_m \mid hah_g^{-1} = a\} \\ &= \frac{n}{\mathrm{vol}(K'_m)} \int_{D^{\times}/\varpi^{\mathbb{Z}}} \mathbf{1}_{hK'_m}(ah_ga^{-1}) da = n O_{h_g}^{D^{\times}/\varpi^{\mathbb{Z}}} \left( \frac{\mathbf{1}_{hK'_m}}{\mathrm{vol}(K'_m)} \right). \end{aligned} \quad \blacksquare$$

Next recall the definition of a transfer of a test function.

**Definition 3.6** For  $\varphi \in \mathcal{H}(G)$  and  $\varphi^D \in \mathcal{H}(D^\times)$ , we say that  $\varphi^D$  is a transfer of  $\varphi$  if

$$\int_{D^\times/\varpi^\mathbb{Z}} \varphi^D(y^{-1}h_g y) dy = (-1)^{n-1} \int_{G/\varpi^\mathbb{Z}} \varphi(x^{-1}gx) dx$$

for every  $g \in G^{\text{ell}}$ .

We know that if  $\varphi \in \mathcal{H}(G)$  is supported on  $G^{\text{ell}}$ , then it has a transfer  $\varphi^D \in \mathcal{H}(D^\times)$  (cf. [Mie11, Lemma 3.2]). The following lemma is obvious:

**Lemma 3.7** Assume that  $\varphi \in \mathcal{H}(G)$  is supported on  $G^{\text{ell}}$  and let  $\varphi^D \in \mathcal{H}(D^\times)$  be its transfer. Put

$$\varphi_\varpi(g) = \sum_{i \in \mathbb{Z}} \varphi(\varpi^i g), \quad \varphi_\varpi^D(h) = \sum_{i \in \mathbb{Z}} \varphi^D(\varpi^i h).$$

Then  $\varphi_\varpi \in \mathcal{H}(G/\varpi^\mathbb{Z})$ ,  $\varphi_\varpi^D \in \mathcal{H}(D^\times/\varpi^\mathbb{Z})$  and  $O_{h_g}^{D^\times/\varpi^\mathbb{Z}}(\varphi_\varpi^D) = (-1)^{n-1} O_g^{G/\varpi^\mathbb{Z}}(\varphi_\varpi)$  for every  $g \in G^{\text{ell}}$ .

**Theorem 3.8** Assume that  $\varphi \in \mathcal{H}(G)$  is supported on  $G^{\text{ell}}$  and let  $\varphi^D \in \mathcal{H}(D^\times)$  be its transfer. Then, for every  $\pi \in \text{Irr}(G/\varpi^\mathbb{Z})$  we have

$$\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) = (-1)^{n-1} n \text{Tr}(\varphi; \pi).$$

*Proof.* Let  $\varphi_\varpi$  and  $\varphi_\varpi^D$  be as in the previous lemma. Then clearly we have

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) &= \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_\varpi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}), \\ &(-1)^{n-1} n \text{Tr}(\varphi; \pi) = (-1)^{n-1} n \text{Tr}(\varphi_\varpi; \pi). \end{aligned}$$

Thus we may replace  $\varphi$  and  $\varphi^D$  by  $\varphi_\varpi$  and  $\varphi_\varpi^D$ , respectively.

Take  $m \geq 0$  such that  $\varphi_\varpi^D$  is  $K'_m$ -invariant, and write

$$\varphi_\varpi^D = \sum_{h \in J} a_h \frac{\mathbf{1}_{hK'_m}}{\text{vol}(K'_m)},$$

where  $J$  is a finite subset of  $D^\times/\varpi^\mathbb{Z}$  and  $a_h \in \mathbb{C}$ . By Corollary 3.2 and Proposition 3.4, we have

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_\varpi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) &= \sum_{i,j \geq 0, h \in J} (-1)^{i+j} a_h \text{Tr}(h; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi)) \\ &= \sum_{h \in J} a_h \int_{G/\varpi^\mathbb{Z}} \eta_{m,h}(g) \theta_\pi(g) dg. \end{aligned}$$

By Theorem 2.4 ii) and Weyl's integral formula (cf. [Kaz86, Theorem F]), we have

$$\int_{G/\varpi^{\mathbb{Z}}} \eta_{m,h}(g) \theta_{\pi}(g) dg = \sum_T \frac{1}{\#W_T} \int_{T^{\text{reg}}/\varpi^{\mathbb{Z}}} D(t) O_t^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) \theta_{\pi}(t) dt,$$

where  $T$  runs through conjugacy classes of elliptic maximal tori of  $G$ ,  $W_T$  denotes the rational Weyl group of  $T$  and  $D(t)$  denotes the Weyl denominator (cf. [Rog83, p. 185]). The measure  $dt$  on  $T/\varpi^{\mathbb{Z}}$  is normalized so that the volume of  $T/\varpi^{\mathbb{Z}}$  is one. Lemma 3.5 and Lemma 3.7 tell us that

$$\begin{aligned} \sum_{h \in J} a_h O_t^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) &= \sum_{h \in J} n a_h O_{h_t}^{D^{\times}/\varpi^{\mathbb{Z}}} \left( \frac{\mathbf{1}_{hK'_m}}{\text{vol}(K'_m)} \right) = n O_{h_t}^{D^{\times}/\varpi^{\mathbb{Z}}}(\varphi_{\varpi}^D) \\ &= (-1)^{n-1} n O_t^{G/\varpi^{\mathbb{Z}}}(\varphi_{\varpi}) \end{aligned}$$

for every  $t \in T^{\text{reg}}$ . By Weyl's integral formula again, we have

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_{\varpi}^D; \text{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) \\ = (-1)^{n-1} n \sum_T \frac{1}{\#W_T} \int_{T^{\text{reg}}/\varpi^{\mathbb{Z}}} D(t) O_t^{G/\varpi^{\mathbb{Z}}}(\varphi_{\varpi}) \theta_{\pi}(t) dt \\ = (-1)^{n-1} n \int_{G/\varpi^{\mathbb{Z}}} \varphi_{\varpi}(g) \theta_{\pi}(g) dg = (-1)^{n-1} n \text{Tr}(\varphi_{\varpi}; \pi), \end{aligned}$$

as desired. ■

## 4 Faltings isomorphism

Here we freely use the notation in [Mie11, Section 2]. We need the following deep theorem due to Faltings ([Fal02], see also [FGL08] for more detailed exposition):

**Theorem 4.1** *We have a  $G \times D^{\times}$ -equivariant isomorphism  $H_{\text{Dr}}^i \cong H_{\text{LT}}^i$  for every  $i$ .*

Note that the proof of Faltings' theorem does not require automorphic method. It gives the following very important finiteness result on  $H_{\text{Dr}}^i$ .

**Corollary 4.2** *The  $G$ -representation  $H_{\text{Dr}}^i$  is admissible.*

*Proof.* Put  $K_m = \text{Ker}(\text{GL}_n(\mathcal{O}) \rightarrow \text{GL}_n(\mathcal{O}/(\varpi^m)))$ . Since

$$(H_{\text{LT}}^i)^{K_m} = H_c^i((M_m/\varpi^{\mathbb{Z}}) \otimes_{\check{F}} \overline{\check{F}}, \overline{\mathbb{Q}}_{\ell})$$

is finite-dimensional,  $H_{\text{LT}}^i$  is an admissible representation of  $G$ . Thus, by Theorem 4.1,  $H_{\text{Dr}}^i$  is also an admissible representation of  $G$ . ■

**Corollary 4.3** For every  $\pi \in \mathbf{Irr}(G/\varpi^{\mathbb{Z}})$  and integers  $i, j \geq 0$ ,  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi)$  is a finite-dimensional smooth representation of  $D^\times$ . Moreover,  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi) = 0$  if  $j \geq n$ .

*Proof.* Let  $\mathfrak{s}$  be the cuspidal support of  $\pi$ , and  $H_{\mathrm{Dr}, \mathfrak{s}}^i$  be the  $\mathfrak{s}$ -component of  $H_{\mathrm{Dr}}^i$ . Clearly we have  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi) = \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}, \mathfrak{s}}^i, \pi)$ .

Let us observe that  $H_{\mathrm{Dr}, \mathfrak{s}}^i$  is a finitely generated  $G$ -representation. Since  $H_{\mathrm{Dr}, \mathfrak{s}}^i$  is an admissible  $G$ -representation by Corollary 4.2, it is  $\mathfrak{Z}(G)$ -admissible, where  $\mathfrak{Z}(G)$  denotes the Bernstein center of  $G$  (cf. [Ber84, §3.1]). Therefore, [Ber84, Corollaire 3.10] tells us that  $H_{\mathrm{Dr}, \mathfrak{s}}^i$  is finitely generated. In particular, there exists a compact open subgroup  $K' \subset D^\times$  which acts on  $H_{\mathrm{Dr}, \mathfrak{s}}^i$  trivially.

Therefore, by [SS97, Corollary II.3.3],  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}, \mathfrak{s}}^i, \pi)$  is finite-dimensional and vanishes if  $j \geq n$ . The natural action of  $D^\times$  on  $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}, \mathfrak{s}}^i, \pi)$  is smooth, since the action of  $K'$  is trivial. This completes the proof.  $\blacksquare$

**Definition 4.4** For  $\pi \in \mathbf{Irr}(G/\varpi^{\mathbb{Z}})$ , put

$$H_{\mathrm{Dr}}[\pi] = \sum_{i, j \geq 0} (-1)^{i+j} \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi) \quad \text{in } R(D^\times/\varpi^{\mathbb{Z}}).$$

We can consider the character  $\theta_{H_{\mathrm{Dr}}[\pi]}$  of  $H_{\mathrm{Dr}}[\pi]$ . Theorem 3.8 can be written in the following way:

**Theorem 4.5** For every  $\pi \in \mathbf{Irr}(G/\varpi^{\mathbb{Z}})$  and  $h \in (D^\times)^{\mathrm{reg}}$ , we have

$$\theta_{H_{\mathrm{Dr}}[\pi]}(h) = n\theta_\pi(g_h).$$

*Proof.* Theorem 3.8 says that  $\mathrm{Tr}(\varphi^D; H_{\mathrm{Dr}}[\pi]) = (-1)^{n-1} n \mathrm{Tr}(\varphi; \pi)$ . We can use exactly the same method as in the proof of [Mie11, Theorem 4.3].  $\blacksquare$

The following is another consequence of Theorem 4.1:

**Corollary 4.6** For every  $\rho \in \mathbf{Irr}(D^\times/\varpi^{\mathbb{Z}})$ ,  $H_{\mathrm{Dr}}^i[\rho] = \mathrm{Hom}_{D^\times}(H_{\mathrm{Dr}}^i, \rho)^{\mathrm{sm}}$  is a smooth  $G$ -representation of finite length. Moreover  $H_{\mathrm{Dr}}^i[\rho]$  is isomorphic to  $\mathrm{Hom}_{D^\times}(\rho, H_{\mathrm{Dr}}^i)^\vee$ .

*Proof.* By Proposition 2.1, Corollary 4.2 and [Mie11, Lemma 5.2],  $H_{\mathrm{Dr}}^i[\rho]$  is a smooth  $G$ -representation of finite length. In the proof of [Mie11, Lemma 5.2], a  $G$ -equivariant injection  $H_{\mathrm{Dr}}^i[\rho] \hookrightarrow \mathrm{Hom}_{D^\times}(\rho, H_{\mathrm{Dr}}^i)^\vee$  is constructed. It is easy to see that it is actually an isomorphism.  $\blacksquare$

**Definition 4.7** For  $\rho \in \mathbf{Irr}(D^\times/\varpi^{\mathbb{Z}})$ , put

$$H_{\mathrm{Dr}}[\rho] = \sum_i (-1)^i H_{\mathrm{Dr}}^i[\rho] \quad \text{in } R(G/\varpi^{\mathbb{Z}}).$$

We can consider the character  $\theta_{H_{\text{Dr}}[\rho]}$  of  $H_{\text{Dr}}[\rho]$ . Corollary 2.5 can be written in the following way:

**Theorem 4.8** *For every  $\rho \in \text{Irr}(D^\times/\varpi^\mathbb{Z})$  and  $h \in (D^\times)^{\text{reg}}$ , we have*

$$\theta_{H_{\text{Dr}}[\rho]}(g_h) = n\theta_\rho(h).$$

*Proof.* We denote the natural homomorphism  $R(G/\varpi^\mathbb{Z}) \longrightarrow K(G/\varpi^\mathbb{Z})$  by EP. By [Dat07, Lemma 3.7], the composite of

$$R(G/\varpi^\mathbb{Z}) \xrightarrow{\text{EP}} K(G/\varpi^\mathbb{Z}) \xrightarrow{\text{Rk}} \overline{\mathcal{H}}(G/\varpi^\mathbb{Z}) \xrightarrow{\vee} C^\infty(G^{\text{ell}})$$

coincides with  $\pi \longmapsto \theta_\pi|_{G^{\text{ell}}}$  (it was originally proved in [SS97, Theorem III.4.23]). Therefore, by Corollary 2.5 and Corollary 4.6 we have

$$\theta_{H_{\text{Dr}}[\rho]}(g) = ((\vee \circ \text{Rk} \circ \text{EP})(H_{\text{Dr}}[\rho]))(g) = n\theta_\rho(h_g)$$

for every  $g \in G^{\text{ell}}$ . Hence  $\theta_{H_{\text{Dr}}[\rho]}(g_h) = n\theta_\rho(h)$  for every  $h \in (D^\times)^{\text{reg}}$ , as desired.  $\blacksquare$

**Remark 4.9** The proof of [SS97, Theorem III.4.23] seems to use [Kaz86, Theorem 0], whose proof relies on global technique. However, Theorem 4.1 and [Mie11, Theorem 4.3] give an alternative proof of Theorem 4.8, which does not involve any global argument.

## 5 Complements on representation theory

For locally constant class functions  $\varphi_1, \varphi_2$  on  $G^{\text{ell}}/\varpi^\mathbb{Z}$ , put

$$\langle \varphi_1, \varphi_2 \rangle_{\text{ell}} = \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) \varphi_1(t) \overline{\varphi_2(t)} dt,$$

where  $T$  runs through conjugacy classes of elliptic maximal tori of  $G$ . Other notations are also the same as in the proof of Theorem 3.8.

For locally constant functions  $\phi_1, \phi_2$  on  $D^\times/\varpi^\mathbb{Z}$ , put

$$\langle \phi_1, \phi_2 \rangle = \int_{D^\times/\varpi^\mathbb{Z}} \phi_1(h) \overline{\phi_2(h)} dh,$$

where the measure  $dh$  is normalized so that the volume of the compact group  $D^\times/\varpi^\mathbb{Z}$  is one.

These two pairings are compatible, in the sense of the following lemma:

**Lemma 5.1** *Let  $\varphi_1, \varphi_2$  be locally constant class functions on  $G^{\text{ell}}/\varpi^\mathbb{Z}$ , and  $\phi_1, \phi_2$  locally constant class functions on  $D^\times/\varpi^\mathbb{Z}$ . Assume that  $\varphi_i(g_h) = \phi_i(h)$  for every  $h \in (D^\times)^{\text{reg}}$ . Then, we have*

$$\langle \varphi_1, \varphi_2 \rangle_{\text{ell}} = \langle \phi_1, \phi_2 \rangle.$$

*Proof.* Clear from Weyl's integral formula for  $D^\times$ . ■

The following orthogonality relation of characters is very important for our work.

**Proposition 5.2** *For  $\pi_1, \pi_2 \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$ , we have*

$$\langle \theta_{\pi_1}, \theta_{\pi_2} \rangle_{\text{ell}} = \begin{cases} 1 & \pi_1 \cong \pi_2, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $\omega_1, \omega_2$  be the central characters of  $\pi_1, \pi_2$ , respectively. Then they are unitary, since  $F^\times/\varpi^\mathbb{Z}$  is compact. If  $\omega_1 = \omega_2$ , then the lemma follows immediately from [Rog83, Lemma 5.3]. Otherwise,

$$\begin{aligned} \int_{T/\varpi^\mathbb{Z}} D(t) \theta_{\pi_1}(t) \overline{\theta_{\pi_2}(t)} dt &= \int_{T/Z_G} \left( \int_{Z_G/\varpi^\mathbb{Z}} D(tz) \theta_{\pi_1}(tz) \overline{\theta_{\pi_2}(tz)} dz \right) dt \\ &= \int_{T/Z_G} \left( \int_{Z_G/\varpi^\mathbb{Z}} \omega_1(z) \overline{\omega_2(z)} dz \right) D(t) \theta_{\pi_1}(t) \overline{\theta_{\pi_2}(t)} dt = 0, \end{aligned}$$

as desired. ■

**Remark 5.3** The proof of [Rog83, Theorem 5.3] (for example, [DKV84, §A.3, §A.4]) seems to need a global argument, such as Howe's conjecture due to Clozel. However, at least if  $n$  is prime, we can give a purely local proof of Proposition 5.2 as follows. Here we use freely the notation which will be introduced in the next section.

Note that, if  $n$  is prime, then any irreducible discrete series representation of  $G$  is either a twisted Steinberg representation or supercuspidal (*cf.* [Zel80, Theorem 9.3]). First assume that both  $\pi_1$  and  $\pi_2$  are twisted Steinberg representations, and write  $\pi_1 = \mathbf{St}_{\chi_1}$  and  $\pi_2 = \mathbf{St}_{\chi_2}$ , where  $\chi_1$  and  $\chi_2$  are characters of  $F^\times/\varpi^\mathbb{Z}$ . Then, by Lemma 5.1 and Lemma 6.7, we have

$$\langle \theta_{\mathbf{St}_{\chi_1}}, \theta_{\mathbf{St}_{\chi_2}} \rangle_{\text{ell}} = \langle \theta_{\chi_1 \circ \text{Nrd}}, \theta_{\chi_2 \circ \text{Nrd}} \rangle = \int_{F^\times/\varpi^\mathbb{Z}} \chi_1(z) \overline{\chi_2(z)} dz = \begin{cases} 1 & \chi_1 = \chi_2, \\ 0 & \chi_1 \neq \chi_2. \end{cases}$$

Since  $\chi_1 \neq \chi_2$  implies  $\mathbf{St}_{\chi_1} \not\cong \mathbf{St}_{\chi_2}$  (their characters are different), we have the orthogonality relation in this case.

Next assume that  $\pi_2$  is supercuspidal. Then, by [DKV84, §A.3.e, §A.3.g], we can find a matrix coefficient  $\phi \in \mathcal{H}(G/\varpi^\mathbb{Z})$  of  $\pi_2$  satisfying the following:

- (a)  $O_g^{G/\varpi^\mathbb{Z}}(\phi) = \overline{\theta_{\pi_2}(g)}$  for  $g \in G^{\text{ell}}$  and  $\int_{Z(g) \backslash G} \phi(x^{-1}gx) dx = 0$  for  $g \in G^{\text{reg}} \setminus G^{\text{ell}}$ .
- (b)  $\text{Tr}(\phi; \pi_2) = 1$  and  $\text{Tr}(\phi; \pi_1) = 0$  for  $\pi_1 \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$  with  $\pi_1 \not\cong \pi_2$ .

By (a) and Weyl's integral formula, we have

$$\begin{aligned} \text{Tr}(\phi; \pi_1) &= \int_{G/\varpi^\mathbb{Z}} \phi(g) \theta_{\pi_1}(g) dg = \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) O_t^{G/\varpi^\mathbb{Z}}(\phi) \theta_{\pi_1}(t) dt \\ &= \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) \overline{\theta_{\pi_2}(t)} \theta_{\pi_1}(t) dt = \langle \theta_{\pi_1}, \theta_{\pi_2} \rangle_{\text{ell}}. \end{aligned}$$

Therefore (b) gives the desired orthogonality relation.

## 6 Local Jacquet-Langlands correspondence

Let  $R_I(G)$  be the submodule of  $R(G)$  generated by the image of parabolically induced representations (cf. [Kaz86]) and put  $\overline{R}(G) = R(G)/R_I(G)$ . It is known that  $\overline{R}(G)$  is a free  $\mathbb{Z}$ -module with a basis  $\{[\pi] \mid \pi \in \mathbf{Disc}(G)\}$  (cf. [Dat07, Lemme 2.1.4]). We regard  $\mathbf{Disc}(G)$  as a subset of  $\overline{R}(G)$ . Recall that the character of a parabolically induced representation vanishes on  $G^{\text{ell}}$ . Therefore,  $\pi \mapsto \theta_\pi|_{G^{\text{ell}}}$  induces a map  $\overline{R}(G) \rightarrow C^\infty(G^{\text{ell}})$ .

Moreover, we denote by  $\overline{R}(G/\varpi^\mathbb{Z})$  the image of  $R(G/\varpi^\mathbb{Z})$  in  $\overline{R}(G)$ . It is easy to see that  $\{[\pi] \mid \pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})\}$  gives a basis of  $\overline{R}(G/\varpi^\mathbb{Z})$ . Set  $\overline{R}(G/\varpi^\mathbb{Z})_{\mathbb{Q}} = \overline{R}(G/\varpi^\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\overline{R}(G)_{\mathbb{Q}} = \overline{R}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

**Lemma 6.1** *The homomorphism  $\widetilde{LJ}: R(G/\varpi^\mathbb{Z}) \rightarrow R(D^\times/\varpi^\mathbb{Z})_{\mathbb{Q}}$  given by*

$$\pi \mapsto \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\pi] \quad \text{for } \pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$$

factors through  $\overline{R}(G/\varpi^\mathbb{Z})$ .

*Proof.* Let  $P$  be a proper parabolic subgroup of  $G$  with Levi factor  $M$  and  $\sigma$  an irreducible smooth representation of  $M$  on which  $\varpi \in Z_M$  acts trivially. By Theorem 4.5, the character of  $\widetilde{LJ}(\text{Ind}_P^G \sigma)$  on  $(D^\times)^{\text{reg}}$  is given by  $h \mapsto (-1)^{n-1} \theta_{\text{Ind}_P^G \sigma}(g_h) = 0$ . Since  $(D^\times)^{\text{reg}}$  is dense in  $D^\times$ , it vanishes for every  $h \in D^\times$ . Thus  $\widetilde{LJ}(\text{Ind}_P^G \sigma) = 0$  by linear independence of characters. Since the kernel of  $R(G/\varpi^\mathbb{Z}) \rightarrow \overline{R}(G/\varpi^\mathbb{Z})$  is generated by such representations as  $\text{Ind}_P^G \sigma$ , we conclude the proof.  $\blacksquare$

The following is the main construction in this paper.

**Definition 6.2** We define two homomorphisms

$$JL: R(D^\times/\varpi^\mathbb{Z})_{\mathbb{Q}} \rightarrow \overline{R}(G/\varpi^\mathbb{Z})_{\mathbb{Q}}, \quad LJ: \overline{R}(G/\varpi^\mathbb{Z})_{\mathbb{Q}} \rightarrow R(D^\times/\varpi^\mathbb{Z})_{\mathbb{Q}}$$

by

$$JL(\rho) = \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\rho], \quad LJ(\pi) = \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\pi]$$

for  $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$  and  $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$ . The latter map is well-defined by the previous lemma.

**Proposition 6.3** i) *We have the character relations*

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every  $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$ ,  $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$  and  $h \in (D^\times)^{\text{reg}}$ .

ii) *For every  $\rho, \rho' \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$  and  $\pi, \pi' \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$ , we have*

$$\begin{aligned} \langle \theta_{JL(\rho)}, \theta_{JL(\rho')} \rangle_{\text{ell}} &= \langle \theta_\rho, \theta_{\rho'} \rangle, \quad \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle = \langle \theta_\pi, \theta_{\pi'} \rangle_{\text{ell}}, \\ \langle \theta_{JL(\rho)}, \theta_\pi \rangle_{\text{ell}} &= \langle \theta_\rho, \theta_{LJ(\pi)} \rangle. \end{aligned}$$

- iii) Two maps  $JL$  and  $LJ$  are inverse to each other.
- iv) The map  $JL$  is compatible with character twists. Namely, for a character  $\chi$  of  $F^\times$  which is trivial on  $\varpi^{n\mathbb{Z}} \subset F^\times$ , we have  $JL(\rho \otimes (\chi \circ \text{Nrd})) = JL(\rho) \otimes (\chi \circ \det)$ . The same holds for  $LJ$ .
- v) The map  $JL$  preserves central characters. Namely, for  $\rho \in \mathbf{Irr}(D^\times / \varpi^\mathbb{Z})$ , write  $JL(\rho) = \sum_{\pi \in \mathbf{Disc}(G / \varpi^\mathbb{Z})} a_\pi [\pi]$ . Then, every  $\pi$  with  $a_\pi \neq 0$  has the same central character as  $\rho$ . The same holds for  $LJ$ .

*Proof.* i) is clear from Theorem 4.5 and Theorem 4.8. ii) follows from i) and Lemma 5.1.

Prove iii). For  $\pi \in \mathbf{Disc}(G / \varpi^\mathbb{Z})$ , write  $JL(LJ(\pi)) = \sum_{\pi' \in \mathbf{Disc}(G / \varpi^\mathbb{Z})} a_{\pi'} [\pi']$ . Then, by ii) and Proposition 5.2 we have

$$a_{\pi'} = \langle \theta_{JL(LJ(\pi))}, \theta_{\pi'} \rangle_{\text{ell}} = \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle = \langle \theta_\pi, \theta_{\pi'} \rangle_{\text{ell}}.$$

Therefore,  $a_{\pi'} = 1$  if  $\pi' = \pi$ , and  $a_{\pi'} = 0$  otherwise. In other words,  $JL(LJ(\pi)) = \pi$ . Thus we have  $JL \circ LJ = \text{id}$ . Similarly we can prove that  $LJ \circ JL = \text{id}$ .

For iv), it suffices to show that  $LJ$  is compatible with character twists. Let  $\pi \in \mathbf{Irr}(G / \varpi^\mathbb{Z})$  and  $\chi$  be a character of  $F^\times$  which is trivial on  $\varpi^{n\mathbb{Z}} \subset F^\times$ . Then, for every  $h \in (D^\times)^{\text{reg}}$ , we have

$$\begin{aligned} \theta_{LJ(\pi \otimes (\chi \circ \det))}(h) &= (-1)^{n-1} \theta_{\pi \otimes (\chi \circ \det)}(g_h) = (-1)^{n-1} \chi(\det g_h) \theta_\pi(g_h) \\ &= \chi(\text{Nrd } h) \theta_{LJ(\pi)}(h) = \theta_{LJ(\pi) \otimes (\chi \circ \text{Nrd})}(h). \end{aligned}$$

Since  $(D^\times)^{\text{reg}}$  is dense in  $D^\times$ , we have  $\theta_{LJ(\pi \otimes (\chi \circ \det))} = \theta_{LJ(\pi) \otimes (\chi \circ \text{Nrd})}$ . By linear independence of characters, we conclude that  $LJ(\pi \otimes (\chi \circ \det)) = LJ(\pi) \otimes (\chi \circ \text{Nrd})$ .

Finally we prove v). Write  $JL(\rho) = \sum_{\pi \in \mathbf{Disc}(G / \varpi^\mathbb{Z})} a_\pi [\pi]$ . By Proposition 5.2,  $a_\pi = \langle \theta_{JL(\rho)}, \theta_\pi \rangle_{\text{ell}}$ . Assume that the central character of  $\pi \in \mathbf{Disc}(G / \varpi^\mathbb{Z})$  is different from that of  $\rho$ . Then, we have

$$a_\pi = \langle \theta_{JL(\rho)}, \theta_\pi \rangle_{\text{ell}} = \sum_T \frac{1}{\#W_T} \int_{T / \varpi^\mathbb{Z}} D(t) \theta_{JL(\rho)}(t) \overline{\theta_\pi(t)} dt = 0$$

in the same way as in the proof of Proposition 5.2. Therefore  $JL$  preserves central characters. By this result and ii), we have  $\langle \theta_\rho, \theta_{LJ(\pi)} \rangle = 0$  unless  $\rho$  and  $\pi$  have the same central character. This means that the coefficient of  $\rho \in \mathbf{Irr}(D^\times / \varpi^\mathbb{Z})$  in  $LJ(\pi)$  is zero unless the central character of  $\rho$  is the same as that of  $\pi$ . Namely,  $LJ$  preserves central characters.  $\blacksquare$

By twisting, we can extend  $JL$  and  $LJ$  to maps between  $R(D^\times)$  and  $\overline{R}(G)$ .

**Proposition 6.4** *There exists a unique extension of  $JL$  to a homomorphism from  $R(D^\times)_{\mathbb{Q}}$  to  $\overline{R}(G)_{\mathbb{Q}}$  which is compatible with character twists. Similarly, we have a unique extension of  $LJ$  to a homomorphism  $\overline{R}(G)_{\mathbb{Q}} \rightarrow R(D^\times)_{\mathbb{Q}}$  which is compatible with character twists. We denote them by  $JL$  and  $LJ$  again. These are inverse to each other, satisfy the same character relations as in Proposition 6.3 i), and preserve central characters.*

*Proof.* For  $\rho \in \mathbf{Irr}(D^\times)$ , let  $\omega_\rho$  be its central character. Take  $c \in \mathbb{C}^\times$  such that  $c^n = \omega_\rho(\varpi)$ , and consider the character  $\chi_c: z \mapsto c^{v_F(z)}$  of  $F^\times$ . Then  $\rho \otimes (\chi_c^{-1} \circ \text{Nrd}) \in \mathbf{Irr}(D^\times / \varpi^\mathbb{Z})$ . Extend  $JL$  to  $R(D^\times)_\mathbb{Q} \rightarrow \overline{R}(G)_\mathbb{Q}$  by

$$JL(\rho) = JL(\rho \otimes (\chi_c^{-1} \circ \text{Nrd})) \otimes (\chi_c \circ \det).$$

By Proposition 6.3 iv), it is independent of the choice of  $c$ . Moreover, we can easily observe that it is the unique extension of the original  $JL$  which is compatible with character twists. Similarly, we can uniquely extend  $LJ$  to a map  $\overline{R}(G)_\mathbb{Q} \rightarrow R(D^\times)_\mathbb{Q}$  compatible with character twists.

By using Proposition 6.3 v), we can easily check that the extended  $JL$  and  $LJ$  are inverse to each other. The remaining parts are also immediate consequences of Proposition 6.3 i), v).  $\blacksquare$

Next we will observe the uniqueness of the maps  $JL$ ,  $LJ$  satisfying the character relations.

**Proposition 6.5** i) Let  $JL': R(D^\times)_\mathbb{Q} \rightarrow \overline{R}(G)_\mathbb{Q}$  be a homomorphism satisfying the character relation  $\theta_\rho(h) = (-1)^{n-1} \theta_{JL'(\rho)}(g_h)$  for every  $h \in (D^\times)^{\text{reg}}$ . Then we have  $JL' = JL$ .  
ii) Let  $LJ': \overline{R}(G)_\mathbb{Q} \rightarrow R(D^\times)_\mathbb{Q}$  be a homomorphism satisfying the character relation  $\theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ'(\pi)}(h)$  for every  $h \in (D^\times)^{\text{reg}}$ . Then we have  $LJ' = LJ$ .

*Proof.* To prove i), it suffices to show that  $LJ \circ JL' = \text{id}$ . By the character relation, we have  $\theta_{LJ(JL'(\rho))}(h) = \theta_\rho(h)$  for every  $\rho \in \mathbf{Irr}(D^\times)$  and  $h \in (D^\times)^{\text{reg}}$ . Thus we can conclude that  $LJ(JL'(\rho)) = \rho$  by linear independence of characters for  $D^\times$ . For ii), prove  $LJ' \circ JL = \text{id}$  by a similar argument.  $\blacksquare$

So far, we have obtained the following theorem:

**Theorem 6.6** We can construct the following two homomorphisms geometrically:

$$JL: R(D^\times)_\mathbb{Q} \rightarrow \overline{R}(G)_\mathbb{Q}, \quad LJ: \overline{R}(G)_\mathbb{Q} \rightarrow R(D^\times)_\mathbb{Q}.$$

These two maps are inverse to each other, and satisfy the character relations

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every  $h \in (D^\times)^{\text{reg}}$ . They are characterized by these character relations. Moreover,  $JL$  and  $LJ$  are compatible with character twists, and preserve central characters.

Let  $B \subset G$  be the Borel subgroup consisting of upper triangular matrices. Recall that the Steinberg representation  $\mathbf{St}$  is the unique irreducible quotient of the unnormalized induction  $\text{Ind}_B^G \mathbf{1}$  from the trivial character  $\mathbf{1}$  on  $B$ . For a character  $\chi$  of  $F^\times$ , put  $\mathbf{St}_\chi = \mathbf{St} \otimes (\chi \circ \det)$ . A representation of the form  $\mathbf{St}_\chi$  is called a twisted Steinberg representation. It is an irreducible discrete series representation of  $G$ . The following lemma is very well-known:

**Lemma 6.7** *We have  $\theta_{\chi \circ \text{Nrd}}(h) = (-1)^{n-1} \theta_{\mathbf{St}_\chi}(g_h)$  for a character  $\chi$  of  $F^\times$  and  $h \in (D^\times)^{\text{reg}}$ .*

*Proof.* In  $\overline{R}(G)$ , we have  $[\mathbf{St}_\chi] = (-1)^{n-1}[\chi \circ \det]$  (cf. [Dat07, Remarque 2.1.14]). As the character of a parabolically induced representation vanishes on  $G^{\text{ell}}$ , we have

$$\begin{aligned} \theta_{\mathbf{St}_\chi}(g_h) &= (-1)^{n-1} \theta_{\chi \circ \det}(g_h) = (-1)^{n-1} \chi(\det g_h) = (-1)^{n-1} \chi(\text{Nrd } h) \\ &= (-1)^{n-1} \theta_{\chi \circ \text{Nrd}}(h), \end{aligned}$$

as desired. ■

**Corollary 6.8** *For a character  $\chi$  of  $F^\times$ , we have  $JL(\chi \circ \text{Nrd}) = \mathbf{St}_\chi$  and  $LJ(\mathbf{St}_\chi) = \chi \circ \text{Nrd}$ .*

*Proof.* By Lemma 6.7, we have  $\theta_{LJ(\mathbf{St}_\chi)} = \theta_{\chi \circ \text{Nrd}}$ . Linear independence of characters tells us that  $LJ(\mathbf{St}_\chi) = \chi \circ \text{Nrd}$ . ■

The following is a consequence of the non-cuspidality result in [Mie10b]:

**Proposition 6.9** *For an irreducible supercuspidal representation  $\pi$  of  $G$ , write  $LJ(\pi) = \sum_{\rho \in \text{Irr}(D^\times)} a_\rho [\rho]$ . Then we have  $a_\rho \geq 0$  for every  $\rho$ .*

*Proof.* We may assume that  $\pi \in \text{Irr}(G/\varpi^\mathbb{Z})$ . Since  $\pi$  is injective in the category of smooth  $G/\varpi^\mathbb{Z}$ -representations,  $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi) = 0$  unless  $j = 0$ . By Theorem 4.1 and [Mie10b, Theorem 3.7], we have  $\text{Hom}_{G/\varpi^\mathbb{Z}}(H_{\text{Dr}}^i, \pi) = 0$  unless  $i = n - 1$ . Therefore we have  $LJ(\pi) = n^{-1}[\text{Hom}_{G/\varpi^\mathbb{Z}}(H_{\text{Dr}}^{n-1}, \pi)]$ . This concludes the proof. ■

Now we can prove the local Jacquet-Langlands correspondence for prime  $n$ .

**Theorem 6.10** *Assume that  $n$  is a prime number. Then  $JL$  induces a bijection*

$$JL: \text{Irr}(D^\times) \xrightarrow{\cong} \text{Disc}(G)$$

*satisfying the character relation  $\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h)$  for every  $h \in (D^\times)^{\text{reg}}$ .*

*Proof.* For simplicity, we denote by  $\mathbf{Cusp}(G/\varpi^\mathbb{Z})$  the subset of  $\text{Disc}(G/\varpi^\mathbb{Z})$  consisting of supercuspidal representations. By Theorem 6.6, it suffices to show the following:

- (a) For  $\rho \in \text{Irr}(D^\times/\varpi^\mathbb{Z})$ ,  $JL(\rho) \in \text{Disc}(G/\varpi^\mathbb{Z})$ .
- (b) For  $\pi \in \text{Disc}(G/\varpi^\mathbb{Z})$ ,  $LJ(\pi) \in \text{Irr}(D^\times/\varpi^\mathbb{Z})$ .

First we shall prove (a). If  $\rho$  is a character, then it follows from Corollary 6.8. Assume that  $\rho$  is not a character, and write  $JL(\rho) = \sum_{\pi \in \mathbf{Disc}(G/\varpi^{\mathbb{Z}})} a_{\pi}[\pi]$ . Then,  $a_{\pi} = 0$  unless  $\pi$  is supercuspidal. Indeed, if  $\pi \notin \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$ , then  $\pi$  is a twisted Steinberg representation  $\mathbf{St}_{\chi}$ , for  $n$  is prime (cf. [Zel80, Theorem 9.3]). By Proposition 5.2, Proposition 6.3 ii) and Corollary 6.8, we have

$$a_{\pi} = a_{\mathbf{St}_{\chi}} = \langle \theta_{JL(\rho)}, \theta_{\mathbf{St}_{\chi}} \rangle_{\text{ell}} = \langle \theta_{\rho}, \theta_{LJ(\mathbf{St}_{\chi})} \rangle = \langle \theta_{\rho}, \theta_{\chi \circ \text{Nrd}} \rangle = 0.$$

Since  $JL(\rho) \neq 0$ , there exists at least one  $\pi \in \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$  satisfying  $a_{\pi} \neq 0$ . Let us observe that such  $\pi$  is unique. Assume that there exist  $\pi, \pi' \in \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$  such that  $a_{\pi}$  and  $a_{\pi'}$  are non-zero. Then, we have  $\langle \theta_{LJ(\pi)}, \theta_{\rho} \rangle = \langle \theta_{\pi}, \theta_{JL(\rho)} \rangle_{\text{ell}} = a_{\pi} \neq 0$ , and similarly  $\langle \theta_{LJ(\pi')}, \theta_{\rho} \rangle \neq 0$ . In other words, if we write

$$LJ(\pi) = \sum_{\varrho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\varrho}[\varrho], \quad LJ(\pi') = \sum_{\varrho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b'_{\varrho}[\varrho],$$

then  $b_{\rho} = a_{\pi}$  and  $b'_{\rho} = a_{\pi'}$  are non-zero. On the other hand, Proposition 6.9 tells us that  $b_{\varrho} \geq 0$  and  $b'_{\varrho} \geq 0$  for every  $\varrho$ . Thus, by Proposition 6.3 ii), we conclude that

$$\langle \theta_{\pi}, \theta_{\pi'} \rangle_{\text{ell}} = \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle = \sum_{\varrho} b_{\varrho} b'_{\varrho} > 0,$$

which is equivalent to  $\pi \cong \pi'$  by Proposition 5.2. Now we have  $JL(\rho) = a_{\pi}[\pi]$  for some  $\pi \in \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$ . Moreover, the argument above tells us that  $a_{\pi} \geq 0$ . By Proposition 5.2 and Proposition 6.3 ii), we have

$$1 = \langle \theta_{\rho}, \theta_{\rho} \rangle = \langle \theta_{JL(\rho)}, \theta_{JL(\rho)} \rangle_{\text{ell}} = a_{\pi}^2 \langle \theta_{\pi}, \theta_{\pi} \rangle_{\text{ell}} = a_{\pi}^2.$$

Hence we conclude that  $a_{\pi} = 1$  and  $JL(\rho) = \pi \in \mathbf{Disc}(G/\varpi^{\mathbb{Z}})$ .

Next prove (b). Write  $LJ(\pi) = \sum_{\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\rho}[\rho]$ . Then, since  $JL$  and  $LJ$  are inverse to each other, we have  $\pi = \sum_{\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\rho} JL(\rho)$ , and thus

$$1 = \sum_{\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\rho} \langle \theta_{JL(\rho)}, \theta_{\pi} \rangle_{\text{ell}}.$$

By (a) and Proposition 5.2, there exists  $\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})$  such that  $\pi = JL(\rho)$ . Then  $LJ(\pi) = \rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})$ , as desired.  $\blacksquare$

## References

[BC91] J.-F. Boutot and H. Carayol, *Uniformisation  $p$ -adique des courbes de Shimura: les théorèmes de Čerednik et de Drinfel'd*, Astérisque (1991), no. 196-197, 7, 45–158 (1992), Courbes modulaires et courbes de Shimura (Orsay, 1987/1988).

- [Ber84] J. N. Bernstein, *Le “centre” de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32.
- [BH00] C. J. Bushnell and G. Henniart, *Correspondance de Jacquet-Langlands explicite. II. Le cas de degré égal à la caractéristique résiduelle*, Manuscripta Math. **102** (2000), no. 2, 211–225.
- [BH05] ———, *Local tame lifting for  $GL(n)$ . III. Explicit base change and Jacquet-Langlands correspondence*, J. Reine Angew. Math. **580** (2005), 39–100.
- [Boy09] P. Boyer, *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, Invent. Math. **177** (2009), no. 2, 239–280.
- [Dat00] J.-F. Dat, *On the  $K_0$  of a  $p$ -adic group*, Invent. Math. **140** (2000), no. 1, 171–226.
- [Dat07] ———, *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Invent. Math. **169** (2007), no. 1, 75–152.
- [Dat11] ———, *Un cas simple de correspondance de Jacquet-Langlands locale modulo  $\ell$* , to appear in Proc. London Math. Soc., 2011.
- [DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples  $p$ -adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 33–117.
- [Dri76] V. G. Drinfeld, *Coverings of  $p$ -adic symmetric domains*, Funkcional. Anal. i Prilozhen. **10** (1976), no. 2, 29–40.
- [Fal94] G. Faltings, *The trace formula and Drinfel'd's upper halfplane*, Duke Math. J. **76** (1994), no. 2, 467–481.
- [Fal02] ———, *A relation between two moduli spaces studied by V. G. Drinfeld*, Algebraic number theory and algebraic geometry, Contemp. Math., vol. 300, Amer. Math. Soc., Providence, RI, 2002, pp. 115–129.
- [Far04] L. Fargues, *Cohomologie des espaces de modules de groupes  $p$ -divisibles et correspondances de Langlands locales*, Astérisque (2004), no. 291, 1–199, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales.
- [FGL08] L. Fargues, A. Genestier, and V. Lafforgue, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in Mathematics, vol. 262, Birkhäuser Verlag, Basel, 2008.
- [Har97] M. Harris, *Supercuspidal representations in the cohomology of Drinfel'd upper half spaces; elaboration of Carayol's program*, Invent. Math. **129** (1997), no. 1, 75–119.
- [Hen93] G. Henniart, *Correspondance de Jacquet-Langlands explicite. I. Le cas modéré de degré premier*, Séminaire de Théorie des Nombres, Paris, 1990–

91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 85–114.

[Hen06] ———, *On the local Langlands and Jacquet-Langlands correspondences*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1171–1182.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Hub96] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Kaz86] D. Kazhdan, *Cuspidal geometry of  $p$ -adic groups*, J. Analyse Math. **47** (1986), 1–36.

[Mie10a] Y. Mieda, *Lefschetz trace formula for open adic spaces*, preprint, arXiv:1011.1720, 2010.

[Mie10b] ———, *Non-cuspidality outside the middle degree of  $\ell$ -adic cohomology of the Lubin-Tate tower*, Adv. Math. **225** (2010), no. 4, 2287–2297.

[Mie11] ———, *Lefschetz trace formula and  $\ell$ -adic cohomology of Lubin-Tate tower*, to appear in Mathematical Research Letters, 2011.

[Rog83] J. D. Rogawski, *Representations of  $\mathrm{GL}(n)$  and division algebras over a  $p$ -adic field*, Duke Math. J. **50** (1983), no. 1, 161–196.

[RZ96] M. Rapoport and Th. Zink, *Period spaces for  $p$ -divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[SS97] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191.

[Str08] M. Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217** (2008), no. 3, 889–951.

[Zel80] A. V. Zelevinsky, *Induced representations of reductive  $\mathfrak{p}$ -adic groups. II. On irreducible representations of  $\mathrm{GL}(n)$* , Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210.